1. *Consider the equation*

$$x'''' + ax = 0 \tag{1}$$

on the real line $\mathbf{R}$*, where* $a \in \mathbf{C}$ *is a constant coefficient.*

(a) Find all values of a *(real or complex) for which all solutions of the equation are bounded in* $\mathbf{R}$*. (Recall that a function* $f\colon \mathbf{R} \to \mathbf{C}$ *is said to be bounded of if there exists* $C \geq 0$ *such that* $|f(t)| \leq C$ *for each* $t \in \mathbf{R}$*.)*

(b) Find all values of a *(real or complex) for which there exists at least one solution* x *of the equation which is bounded in* $\mathbf{R}$ *but does not vanish identically.*

Solution:

The characteristic polynomial of the equation is

$$\lambda^4 + a = 0\,. \tag{2}$$

Assume first that $a \neq 0$. Then equation (2) has four roots

$$\lambda_1 = \alpha\,,\ \lambda_2 = i\alpha\,,\ \lambda_3 = i^2\alpha\,,\ \text{ and }\ \lambda_4 = i^3\alpha\,, \tag{3}$$

where α is any number with $\alpha^4 + a = 0$. The general solution of (1) is

$$x(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} + C_3e^{\lambda_3 t} + C_4e^{\lambda_4 t}\,. \tag{4}$$

If all solutions are bounded, then so must be the four functions $e^{\lambda_j t}$, $j = 1, 2, 3, 4$. This means that $\operatorname{Re}\lambda_j = 0\,,\ j = 1, 2, 3, 4$. However, the four numbers λ_j form a square in the complex plane and hence cannot all lie on the imaginary line unless $\alpha = 0$ in which case also $a = 0$. Therefore for no $a \neq 0$ the equation can satisfy (a). To satisfy (b) at least one of the λ_j must be on the imaginary axis. But then $\lambda_j^4 > 0$ and hence $a = -\lambda_j^4 < 0$. Vice versa, if $a < 0$ then the root $i\sqrt[4]{-a}$ is on the imaginary axis. We see that (b) is satisfied for $a \neq 0$ if and only if $a < 0$.

It remains to settle the case $a = 0$. In this case the general solution is

$$C_1 + C_2t + C_3t^2 + C_4t^3 \tag{5}$$

and we see that there are unbounded solutions (such as $x(t) = t$) as well as bounded solutions which do not vanish identically (such as $x(t) = 1$).

Summarizing, we see that there is no $a \in \mathbf{C}$ for which all solutions are bounded, and bounded solutions which do not vanish identically exist if and only if $a \leq 0$.

2. *Let* $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ *be a solution of the system*

$$\begin{aligned} x_1' &= -2x_1 + x_3\,, \\ x_2' &= -2x_2 + x_1\,, \\ x_3' &= -2x_3 + x_2\,, \end{aligned}$$

with $x_1(0) = 1\,,\ x_2(0) = 2\,,$ *and* $x_3(0) = 3$*. Find* $\lim_{t\to\infty} x(t)$*.*

Solution:
The characteristic polynomial of the system is

$$p(\lambda) = \det \begin{pmatrix} -2-\lambda & 0 & 1 \\ 1 & -2-\lambda & 0 \\ 0 & 1 & -2-\lambda \end{pmatrix} = (-2-\lambda)^3 + 1\,. \tag{6}$$

The equation $p(\lambda) = 0$ can be rewritten as $(2+\lambda)^3 = 1$ which means that we have three roots λ_j given by

$$\lambda_j = -2 + \zeta_j\ , \quad j = 1, 2, 3\,, \tag{7}$$

where ζ_j are the three roots of the equation $z^3 = 1$. The ζ_j all lie on the unit circle, and hence

$$\operatorname{Re} \lambda_j \le -1\,, \quad j = 1, 2, 3\,. \tag{8}$$

Therefore all solutions of our system must approach zero as $t \to \infty$. This applies also to our particular solution, and hence

$$\lim_{t\to\infty} x(t) = 0\,. \tag{9}$$

3. *Let*

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}.$$

Compute the matrix e^{tA}.

Solution:
We note the matrix is symmetric and therefore it must become diagonal in a suitable orthogonal basis. The characteristic polynomial is $p(\lambda) = \lambda(\lambda+5)$, the eigenvalues are $\lambda_1 = 0, \lambda_2 = -5$, the corresponding eigenvectors are respectively $x^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $x^{(2)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. The transition matrix mapping the canonical basis to the basis $x^{(1)}$, $x^{(2)}$ is the matrix $P = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, its inverse is $P^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$, and we have

$$A = P \begin{pmatrix} 0 & 0 \\ 0 & -5 \end{pmatrix} P^{-1}\,. \tag{10}$$

Therefore

$$e^{tA} = P \begin{pmatrix} 1 & 0 \\ 0 & e^{-5t} \end{pmatrix} P^{-1} = \frac{1}{5} \begin{pmatrix} 4 + e^{-5t} & 2 - 2e^{-5t} \\ 2 - 2e^{-5t} & 1 + 4e^{-5t} \end{pmatrix}. \tag{11}$$